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Improper Integrals, Simple Integrals, and Numerical Quadrature

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The improper Riemann integral $\int_0^\infty f(x)\,dx$ is defined in the simplest case as a limit of proper Riemann integrals, which are themselves limits of Riemann sums. In this paper we discuss the representation of the improper integral as a limit of Riemann sums. Our main result—Theorem 3—gives a condition on the integrand that is necessary and sufficient for such representation of the integral, with the largest natural class of Riemann sums.

Some of the motivation for this paper comes from the theory of numerical integration. Most formulas for numerical quadrature—Simpson's rule, the trapezoid rule, and the Gauss-Legendre formulas, for example—approximate the integral by calculating carefully chosen Riemann sums.* Thus it is the Riemann concept of the integral that is most appropriate for numerical analysis. The quadrature rules mentioned converge to the integral whenever the function being integrated is properly Riemann-integrable; there seems to be no larger class of bounded functions for which quadrature rules converge.

In the case of the improper Riemann integral, the connection with numerical quadrature is obscured by the double limiting process involved. If we wish to use a sequence of quadrature formulas, or quadrature rule,

$$Q_n(f) = \sum_{r=1}^{n} a_{r,n} f(x_{r,n}) \approx \int_0^{T_n} f(x)\,dx, \tag{1}$$

(with T_n approaching infinity as n does) to approximate $\int_0^\infty f$, we find that there is indeed no such sequence which will converge for all improperly Riemann-integrable f's. (A proof of this is given in the Appendix.) However,

* That the Gauss-Legendre formulas define Riemann sums was shown by T. J. Stieltjes in [1].

we may specify characteristics of f which will guarantee convergence for some classes of sequences (1).

(Of course, there is another numerical approach to $\int_0^\infty f(x)\,dx$. If $f(x)$ decreases rapidly as $x \to \infty$, it may be natural to write $f = \omega g$, where ω is some non-negative "weight function" which decreases to zero so rapidly that all its moments on $[0, \infty)$ are finite. One can then use formulas of the form

$$\int_0^\infty \omega(x)\, h(x)\, dx \approx \sum_{r=1}^{n} a_r h(x_r)$$

which are designed to be exact when h is any polynomial of a certain degree. Convergence can then be discussed in terms of the approximation of g by polynomials. The Gauss-Laguerre formulas for $\omega(x) = e^{-x}$ are the most famous of this kind (see e.g. [2]). We will not consider this approach in this paper.)

Let IR denote the class of all real-valued functions f that are defined on $[0, \infty)$, are properly Riemann-integrable over $[0, b]$ whenever $0 \leqslant b < \infty$, and have a (finite) improper Riemann integral

$$\int_0^\infty f(x)\, dx. \tag{2}$$

(In this paper all functions will be taken to be defined on $[0, \infty)$ and to be real-valued[1], unless there is a statement to the contrary.) For any f in IR, there are some sequences of Riemann sums which converge to (2).

In the first section of this paper we shall briefly study the connection between this fact and the question of convergence of numerical quadrature formulas. In the second section we shall determine the class of functions $f \in IR$ for which all sequences of Riemann sums satisfying a natural condition converge to the integral (2); the characterization of this class is in terms of a concept related to the concept of bounded variation, but specifically appropriate to infinite intervals.

1. Regulating Functions

We first set down the following notations, most of which are standard:

Definition. If $0 \leqslant a < b < \infty$, a sequence $x_0 < x_1 < x_2 < \cdots < x_n$ with $x_0 = a$ and $x_n = b$ is called a "partition" of $[a, b]$. If "Π" is the name of the partition, the numbers $x_0, x_1, \ldots, x_n$ will be called the "points" of Π, and be said to "belong to" Π. The quantity $\max_{1 \leqslant r \leqslant n} (x_r - x_{r-1})$ will be

[1] The extension to complex-valued functions is immediate.

called the "gauge" of Π, and be denoted "$|\Pi|$". The intervals $[x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$, will be referred to as "subintervals of Π." Every Riemann sum

$$\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1})$$

with $x_{r-1} \leqslant \xi_r \leqslant x_r$, $r = 1, 2, \ldots, n$, will be said to be "based on Π."

DEFINITION. If a and b are as above, and f is a function bounded on $[a, b]$ and, for $r = 1, 2, \ldots, n$,

$$M_r = \sup_{x_{r-1} \leqslant x \leqslant x_r} f(x), \qquad m_r = \inf_{x_{r-1} \leqslant x \leqslant x_r} f(x),$$

then the sums

$$\sum_{r=1}^{n} M_r(x_r - x_{r-1}), \quad \sum_{r=1}^{n} m_r(x_r - x_{r-1}), \quad \text{and} \quad \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$$

will be called the "upper sum," "lower sum," and "oscillation sum," respectively, of f, based on Π. They will be abbreviated U.S.(f, Π), L.S.(f, Π), and O.S.(f, Π).

DEFINITION. A real-valued function g, defined and strictly decreasing on $(0, b)$ for some $b > 0$, and having $\lim_{x \to 0+} g(x) = +\infty$ is called a "regulating function."

DEFINITION. A function f is said to be "regulated by g" if g is a regulating function and there is a number I satisfying the following condition: For any monotone nondecreasing sequence of positive numbers b_n, with $\lim_{n\to\infty} b_n = \infty$, any sequence of partitions Π_n of $[0, b_n]$ having $|\Pi_n| \to 0$ and $b_n = o(g(|\Pi_n|))$ as $n \to \infty$, and any sequence of Riemann sums Σ_n based on Π_n,

$$\lim_{n\to\infty} \Sigma_n = I. \tag{3}$$

It is easy to see that every f in IR is regulated by some regulating function g and that, conversely, any f that is regulated by some g is in IR.

EXAMPLES. a. If $f(x) = (\sin x)/x$ and Π: $0 = x_0 < x_1 < \cdots < x_n = b$ is a partition, and $x_{r-1} \leqslant \xi_r \leqslant x_r$, then

$$\left| f(\xi_r)(x_r - x_{r-1}) - \int_{x_{r-1}}^{x_r} f(x)\, dx \right| \leqslant (x_r - x_{r-1})\, 0_r$$

where 0_r is the oscillation (the maximum minus the minimum) of f on $[x_{r-1}, x_r]$. If μ_r is the maximum of $|f'(x)|$ on $[x_{r-1}, x_r]$, $0_r \leqslant (x_r - x_{r-1})\, \mu_r$. Now μ_r is less than an absolute constant C_1 for $x_{r-1} \leqslant 1$, and less than C_2/x_{r-1} for $x_{r-1} > 1$. Any Riemann sum for f based on the partition Π differs from $\int_0^b$ by no more than

$$\sum_r (x_r - x_{r-1})^2 \mu_r .$$

This can be written

$$\sum_{x_{r-1}\leqslant 1} (x_r - x_{r-1})^2 \mu_r + \sum_{x_{r-1}>1} (x_r - x_{r-1})^2 \mu_r ,$$

which is less than or equal to

$$C_1 \,|\, \Pi \,|\, \sum_{x_{r-1}\leqslant 1} (x_r - x_{r-1}) + C_2 \,|\, \Pi \,|\, \sum_{x_{r-1}>1} \frac{x_r - x_{r-1}}{x_{r-1}} .$$

The first of these sums is no greater than $1 + |\, \Pi \,|$. The second is

$$\begin{aligned}
\sum_{x_{r-1}>1} \frac{x_r}{x_{r-1}} \frac{x_r - x_{r-1}}{x_r} &\leqslant (1 + |\, \Pi \,|) \sum_{x_{r-1}>1} \frac{x_r - x_{r-1}}{x_r} \\
&\leqslant (1 + |\, \Pi \,|) \sum_{x_{r-1}>1} \log \frac{x_r}{x_{r-1}} \\
&\leqslant (1 + |\, \Pi \,|) \log b.
\end{aligned}$$

Thus the total is less than or equal to

$$(C_1 + C_2)(1 + |\, \Pi \,|)\,|\, \Pi \,|\, \log b. \tag{4}$$

For this to go to zero as b approaches infinity, it is sufficient that $|\, \Pi \,| \log b$ go to zero. It follows that $(\sin x)/x$ is regulated by $g(x) = e^{1/x}$.

b. If $f \in Lip^{\alpha}[0, \infty)$, $0 < \alpha \leqslant 1$, f is regulated by $g(x) = x^{-\alpha}$.

c. As we shall see below, $(\sin x^2)/x^2$ is regulated by every regulating function g.

A quadrature formula (assumed applied to the interval [0, 1] for convenience)

$$Q(f) = \sum_{r=1}^{n} a_r f(x_r) \approx \int_0^1 f(x)\, dx \tag{5}$$

is a Riemann sum for the integral it is approximating when $a_1 + a_2 + \cdots + a_n = 1$ and:

$$1.) \quad a_r > 0, \qquad r = 1, 2, \ldots, n,$$

and, taking $a_0 = 0$,

$$2.) \quad a_0 + a_1 + \cdots + a_{r-1} \leqslant x_r \leqslant a_0 + a_1 + \cdots + a_r$$

for every r. The partition involved is $0 = t_0 < t_1 < \cdots < t_n = 1$, where $t_1 = a_1$, $t_2 = a_1 + a_2$, etc.; the largest a_r is the gauge of the partition. For the n-subinterval trapezoid rule

$$T_n(f) = \frac{1}{2n}(f(0) + f(1)) + \frac{1}{n}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right),$$

and for the n-subinterval Simpson's rule

$$S_n(f) = \frac{1}{6n}(f(0) + f(1)) + \frac{4}{6n}\sum_{r=1}^{n-1} f\left(\frac{2r-1}{2n}\right) + \frac{2}{6n}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right)$$

it is clear that 1.) and 2.) hold, and the gauges are $1/n$ and $2/(3n)$, respectively. For the n-point Gauss-Legendre formula it is well known that 1.) holds and it was shown by Stieltjes [1] that 2.) holds, the largest value of a_r being asymptotic to π/n ([4], p. 350). When any of these rules is applied to an interval $[0, b]$ instead of $[0, 1]$, the gauge of the partition involved is multiplied by b. These facts, and the definition of regulating function, immediately give us:

THEOREM 1. *Let $Q_1, Q_2, \ldots$ be either the sequence $T_1, T_2, \ldots$ of trapezoid rule formulas, or the Simpson sequence $S_1, S_2, \ldots$, or the sequence of* 1*-point,* 2*-point,... Gauss-Legendre formulas. If f is regulated by g, and $Q_n(f, b_n)$ is the result of applying Q_n to integration of f over* $[0, b_n]$, *then*

$$\lim_{n\to\infty} Q_n(f, b_n) = \int_0^\infty f \qquad \text{if} \quad b_n \to \infty, \quad b_n = o(n),$$

and

$$b_n = o\left(g\left(\frac{cb_n}{n}\right)\right). \tag{6}$$

Here $c = 1$ in the trapezoid rule case, $c = 2/3$ for Simpson's rule, and c is any number greater than π for the Gauss-Legendre case.

Similar theorems can easily be found for other sequences of quadrature formulas.

EXAMPLES. a. $f(x) = (\sin x)/x$. For any of the quadrature sequences mentioned in Theorem 1, convergence is assured by having $b_n \log b_n = o(n)$ or $b_n = o(n/\log n)$.

b. If $f \in Lip^\alpha$ $[0, \infty)$, then a sufficient condition for convergence (for the same quadrature sequences) is $b_n = o(n^{\alpha/(1+\alpha)})$.

c. For $f(x) = (\sin x^2)/x^2$, a sufficient condition is $b_n = o(n)$; see below.

2. THE SIMPLE INTEGRAL

For some f the use of regulating functions is unnecessary. We now proceed to characterize these.

DEFINITION. A function f is called "simply integrable over $[0, \infty)$" if there is a number I such that: For any $\epsilon > 0$ there are positive numbers $B = B(\epsilon)$ and $\Delta = \Delta(\epsilon)$ such that if $b > B$ and Π is a partition of $[0, b]$ with $|\Pi| < \Delta$ and Σ is a Riemann sum for f, based on Π, then $|\Sigma - I| < \epsilon$.

In other words, f is simply integrable if the Riemann sums associated with partitions Π of $[0, b]$ approach a unique (finite) limit as long as $b \to \infty$ and $|\Pi| \to 0$ simultaneously.

It is easy to see that if f is simply integrable then $f \in IR$ and the number I is just $\int_0^\infty f$.

Let $Q_1, Q_2, \ldots$ be any sequence of quadrature formulas which are Riemann sums, with largest coefficients $a_1, a_2, \ldots$ when applied on the interval $[0, 1]$ or $[-1, 1]$. If f is simply integrable, $\lim_{n\to\infty} Q_n(f, b_n) = \int_0^\infty f$ whenever $b_n \to \infty$ and $a_n b_n \to 0$. In particular, if $a_n = O(1/n)$, as is the case for the formulas mentioned in Theorem 1, condition (6) of the Theorem can be omitted for simply integrable f.

If $f \in IR$ and is of bounded variation on $[0, \infty)$, (i.e., the total variations of f on all finite intervals $[0, b]$ have a finite least upper bound $V(f)$), it is simply integrable. Given $\epsilon > 0$ one can choose B so large that $|\int_b^\infty f| < \epsilon/2$ for all $b > B$, and Δ so small that $\Delta \cdot V(f) < \epsilon/4$. Set $I = \int_0^\infty f$. If Π: $0 = x_0 < x_1 < \cdots < x_n = b$ is a partition with $b > B$ and $|\Pi| < \Delta$, and Σ is any Riemann sum for f based on Π, then

$$|\Sigma - I| \leqslant \left|\Sigma - \int_0^b f\right| + \left|\int_0^b f - I\right|.$$

The second quantity on the right is $<\epsilon/2$; the first is no more than O.S.(f, Π) which is

$$\leqslant 2 \sum_{r=1}^{n} (f(\alpha_r) - f(\beta_r))(x_r - x_{r-1}),$$

where α_r and β_r are points of $[x_{r-1}, x_r]$ chosen so as to make $f(\alpha_r) - f(\beta_r) \geqslant (M_r - m_r)/2$. Let $t_0, t_1, \ldots, t_m$ be the points $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n$ rearranged in natural order. Then

$$\begin{aligned} |\Sigma - I| &< \epsilon/2 + 2\Delta \sum_{r=1}^{n} |f(\alpha_r) - f(\beta_r)| \\ &< \epsilon/2 + 2\Delta \sum_{s=1}^{m} |f(t_s) - f(t_{s-1})| \\ &< \epsilon/2 + 2\Delta V(f) < \epsilon. \end{aligned}$$

We remark that the simple integral is not an absolute integral. An example is the function f defined as follows: Set

$$t_0 = 0 \qquad \text{and} \qquad t_n = \sum_{r=1}^{n} \frac{2^{r-1}}{r}, \qquad n = 1, 2, \ldots;$$

and set $f(x) = (-1)^{n-1}/2^{n-1}$ on $[t_{n-1}, t_n)$. f is of bounded variation and in IR but $|f|$ is not in IR.

We give a first characterization of simple integrability in terms of an analogue of the classical Riemann Condition for integrability over finite intervals (see, e.g., [3], p. 281).

DEFINITION. A function f is said to satisfy the "Uniform Riemann Condition" when

1.) For every $\epsilon > 0$ there is a $\Delta = \Delta(f, \epsilon)$ such that: if Π is a partition of any finite interval $[0, b]$, of gauge less than Δ, then O.S.$(f, \Pi) < \epsilon$.

2.) For every $\epsilon > 0$ there is a $B = B(f, \epsilon)$ and a $\delta = \delta(f, \epsilon)$ such that: whenever $b' > b > B$ and Π is a partition of $[b, b']$ of gauge less than δ, and Σ is any Riemann sum for f, based on Π, then $|\Sigma| < \epsilon$.

If we were to weaken this definition by permitting Δ to depend on b, and δ to depend on b and b', we would obtain a necessary and sufficient condition for f to be in IR. For then 1.) would become the classical Riemann Condition for the interval $[0, b]$, asserted for every positive b; and 2.) would become equivalent to the statement that

$$\sup_{b'>b>B} \left| \int_b^{b'} f \right|$$

approaches zero as B approaches infinity.

We note that for $f \in IR$, part 1.) of the Uniform Riemann Condition implies part 2.). For, given $\epsilon > 0$, we can choose B so that

$$\left| \int_b^{b'} f \right| < \epsilon/2$$

whenever $b' > b > B$. Setting δ equal to the $\Delta(f, \epsilon/2)$ of part 1.), it follows that if $b' > b > B$ and Π is any partition of $[b, b']$ of gauge less than δ, then O.S.$(f, \Pi) < \epsilon/2$. Then if Σ is any Riemann sum for f, based on Π, it differs from $\int_b^{b'} f$ by less than $\epsilon/2$, and so $|\Sigma| < \epsilon$.

THEOREM 2. *A function f is simply integrable if and only if it satisfies the Uniform Riemann Condition.*

Proof. Only if: Given $\epsilon > 0$, let $B(\epsilon/2)$ and $\Delta(\epsilon/2)$ be the quantities specified in the definition of simple integrability. Since f is Riemann integrable on $[0, B]$, there is a $\delta > 0$ such that if Π is any partition of $[0, B]$ of gauge $<\delta$, O.S.$(f, \Pi) < \epsilon$. It follows that if Π' is any partition of $[0, b]$ of gauge $<\delta$, where $b \leqslant B$, O.S.$(f, \Pi) < \epsilon$. Set $\Delta(f, \epsilon) = \min(\Delta(\epsilon/2), \delta)$. Then if Π is a partition of any interval $[0, b]$ with gauge $\leqslant \Delta(f, \epsilon)$, O.S.$(f, \Pi) < \epsilon$ when $b \leqslant B$. When $b > B$, every Riemann sum associated with Π differs from $\int_0^\infty f$ by less than $\epsilon/2$. Since U.S.(f, Π) is the sup of all such sums and L.S.(f, Π) the inf, O.S.$(f, \Pi) =$ U.S.$(f, \Pi) -$ L.S.$(f, \Pi) < \epsilon$. This establishes part 1.) of the Uniform Riemann Condition. Part 2.) follows from the fact that $f \in IR$.

If: As remarked earlier, the Uniform Riemann Condition implies that $f \in IR$; set $I = \int_0^\infty f$. Given $\epsilon > 0$, choose $B(f, \epsilon/3)$ and $\Delta(f, \epsilon/3)$ as in the definition of the Condition (with $\Delta(f, \epsilon/3) < 1$) and set $B = B(f, \epsilon/3) + 1$. If $b > B$ and $\Pi\colon 0 = x_0 < x_1 < \cdots < x_n = b$ is a partition with $|\Pi| < \Delta(f, \epsilon/3)$ and Σ is any Riemann sum based on Π, let m be the greatest integer such that $x_m \leqslant B$. Let Π_1 be the partition of $[0, x_m]$ by the points $x_0, x_1, \ldots, x_m$, and let Σ_1 be the sum of the first m terms of Σ. Since $x_m > B(f, \epsilon/3)$, the second part of the Uniform Riemann Condition implies that $|I - \int_0^{x_m} f| < \epsilon/3$, and also that $|\Sigma - \Sigma_1| < \epsilon/3$. The first part implies that $|\Sigma_1 - \int_0^{x_m} f| < \epsilon/3$, and so $|\Sigma - I| < \epsilon$, proving the Theorem.

3. BOUNDED COARSE VARIATION

We have noted that bounded variation is a sufficient condition for an improperly integrable function to be simply integrable. It cannot be necessary; from the discussion of the Uniform Riemann Condition it is clear that the difference between improper integrability and simple integrability involves only the behavior of the integrand $f(x)$ as x approaches infinity —and bounded variation involves restrictions on its behavior in finite intervals. We need a different property of functions, appropriate to the infinite interval.

DEFINITION. If ϵ is a positive number, a set of real numbers is called "ϵ-separated" when every two numbers in the set differ by ϵ or more. A partition is ϵ-separated if the set of its points is ϵ-separated.

DEFINITION. If f is a function and $\{x_0, x_1, x_2, ...\}$ is a finite or infinite, strictly increasing, sequence of non-negative numbers, then the (finite or infinite) quantity

$$\sum_i |f(x_i) - f(x_{i-1})| \tag{7}$$

is called "the variation of f on the sequence $\{x_0, x_1, ...\}$". If S is a set of non-negative real numbers with no finite limit point, and S^* the sequence consisting of the elements of S in their natural order, then the "variation of f on S" is just the variation of f on S^*.

DEFINITION. For any function f and any $\epsilon > 0$, the "ϵ-variation of f" (denoted "$V_\epsilon(f)$") is the supremum of the variations of f on all ϵ-separated sets of non-negative real numbers.

DEFINITION. A function f is said to be of "bounded coarse variation" ("BCV") if it has a finite ϵ-variation for every $\epsilon > 0$. The set of all functions of bounded coarse variation will also be denoted "BCV."

If the ϵ-variation of f were finite for every $\epsilon > 0$ and also bounded in ϵ, f would be of BV on $[0, \infty)$. BCV is a weaker condition. It is useful only for infinite intervals—on a finite interval BCV is equivalent to boundedness.

EXAMPLES. a.) $(\sin x)/x$ is not of BCV: The sequence $\pi/2, 3\pi/2, 5\pi/2, ...$ is π-separated and the variation of the function on it is

$$\frac{2}{\pi} + \frac{4}{3\pi} + \frac{4}{5\pi} + \frac{4}{7\pi} + \cdots$$

which is infinite.

b.) If g is positive, monotone decreasing on $[0, \infty)$, and in IR, and $|f| \leqslant g$, then $f \in$ BCV: If a sequence $\{x_0, x_1, ...\}$ is ϵ-separated, $|f(x_i) - f(x_{i-1})| < 2g(x_{i-1})$ for each i, and so the variation of f on the sequence is

$$\leqslant 2g(x_0) + 2g(x_0 + \epsilon) + \cdots \leqslant 2g(0) + 2g(\epsilon) + 2g(2\epsilon) + 2g(3\epsilon) + \cdots ;$$

and the last series converges. Thus $(\sin x^2)/x^2$, which is not of BV, if of BCV.

The following lemma is the basic tool in the proof of Theorem 3.

LEMMA 1. *If Π is a partition of an interval $[a, b]$ of gauge δ or less, and $b - a \geqslant 12\delta$, and f is a real-valued function defined and bounded on $[a, b]$, then there is a δ-separated sequence $t_0 < t_1 < \cdots < t_n$ of points of $[a, b]$ such that the variation of f on the sequence is greater than or equal to*

$$\frac{1}{24}\frac{\text{O.S.}(f, \Pi)}{\delta}.$$

Proof. We first delete points from Π to get a coarser partition Π' which is δ-separated and has $|\Pi'| \leqslant 3\delta$. If the points of Π are $a = x_0 < x_1 < \cdots < x_m = b$, we can do this by setting y_0 equal to x_0, y_1 equal to the least x_r that is greater than or equal to $y_0 + \delta$, y_2 equal to the least x_r that is greater than or equal to $y_1 + \delta$, etc.; stopping when we obtain a y that is greater than $b - \delta$, and substituting b for that y. Say the points of Π' are $a = y_0 < y_1 < \cdots < y_p$ (note that $p \geqslant 4$). If we set $A = \text{O.S.}(f, \Pi)$, then $\text{O.S.}(f, \Pi') \geqslant A$. In each interval $[y_{i-1}, y_i]$ choose points α_i and β_i to satisfy

$$f(\alpha_i) \geqslant M_i - A/(12p\delta), \qquad f(\beta_i) \leqslant m_i + A/(12p\delta)$$

where M_i and m_i are the sup and inf, respectively, of f on $[y_{i-1}, y_i]$. Then

$$\sum_{i=1}^{p} (f(\alpha_i) - f(\beta_i))(y_i - y_{i-1}) \geqslant \frac{A}{2}. \tag{9}$$

If we separate the sum in (9) into the sum over odd values of i and the sum over even values of i, at least one of these two sums must be $\geqslant A/4$; let us say it is the first. Since $|y_i - y_{i-1}| \leqslant 3\delta$, we have

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{p} (f(\alpha_i) - f(\beta_i)) \geqslant \frac{A}{12\delta}; \tag{10}$$

this sum has at least two terms. We will now choose one of α_i and β_i, for each i, and rename it $t_{(i-1)/2}$, so as to have

$$\sum_{r} |f(t_r) - f(t_{r-1})| \geqslant \frac{A}{24\delta}.$$

Since any two consecutive t's will be separated by an interval $[y_{i-1}, y_i]$ (with i even) of length δ or more, this will complete the proof. For $i = 1$ and 3, we make the choice by setting either $t_0 = \alpha_1$ and $t_1 = \beta_3$ or $t_0 = \beta_1$ and $t_1 = \alpha_3$ according as $|f(\alpha_1) - f(\beta_3)|$ or $|f(\beta_1) - f(\alpha_3)|$ is the larger.

Clearly the larger of those two quantities is no less than half their sum, which is $(f(\alpha_1) - f(\beta_1)) + (f(\alpha_2) - f(\beta_2))$. For $i > 3$, we choose for $t_{(i-1)/2}$ either α_i or β_i according as $f(\alpha_i)$ or $f(\beta_i)$ differs by more (in absolute value) from $f(t_{(i-3)/2})$; one of them must differ by at least $(f(\alpha_i) - f(\beta_i))/2$.

The next lemma sharpens our knowledge of the meaning of BCV:

LEMMA 2. *If $f \in$ BCV then for every $\epsilon > 0$ there exist positive numbers $\delta = \delta(f, \epsilon)$ and $B = B(f, \epsilon)$ with the property that whenever $0 < \delta' \leqslant \delta$ and S is a δ'-separated set of points in $[B, \infty)$, the variation of f on S is less than ϵ/δ'.*

Proof. Assume the conclusion is false. Then there is an $\epsilon > 0$, a sequence $\delta_1, \delta_2, \ldots$ of positive numbers decreasing to zero, and a sequence $B_1, B_2, \ldots$ of positive numbers increasing to infinity, such that: For each positive integer i there exists a δ_i-separated sequence $x_{i,0} < x_{i,1} < \cdots < x_{i,k(i)}$ in $[B_i, \infty)$ with

$$\sum_{r=1}^{k(i)} |f(x_{i,r}) - f(x_{i,r-1})| \geqslant \frac{\epsilon}{\delta_i}. \tag{11}$$

By choosing a subsequence of the i's, if necessary, we may arrange that

$$B_i > x_{i-1,k(i-1)} + \delta_1. \tag{12}$$

We shall assume that (12) holds. If ever $x_{i,j} - x_{i,j-1}$ were greater than $2\delta_i$, we could insert additional points $x_{i,r}$ midway between those already present until this were no longer the case, without disturbing (11); so we shall further assume that

$$\delta_i \leqslant x_{i,j} - x_{i,j-1} \leqslant 2\delta_i \tag{13}$$

for all i and j.

Since $f \in$ BCV, $|f|$ is bounded—say by K. Then

$$2Kk(i) \geqslant \sum_{r=1}^{k(i)} |f(x_{i,r}) - f(x_{i,r-1})| \geqslant \frac{\epsilon}{\delta_i}$$

and so

$$k(i) \geqslant \frac{\epsilon}{2K\delta_i}$$

and

$$x_{i,k(i)} - x_{i,0} \geqslant \delta_i k(i) \geqslant \frac{\epsilon}{2K}. \tag{14}$$

Choose a number δ that is less than δ_1 and less than $\epsilon/(24K)$, and choose N so large that $\delta_i < \delta/2$ for $i \geqslant N$. Then, for $i \geqslant N$, the points $x_{i,0}, x_{i,1}, \ldots, x_{i,k(i)}$

constitute a partition Π_i of $[x_{i,0}, x_{i,k(i)}]$ whose gauge is δ or less, while the length of the partitioned interval is not less than 12δ. By Lemma 1 there is a δ-separated sequence S_i of points in $[x_{i,0}, x_{i,k(i)}]$ such that the variation of f on the sequence is not less than

$$\frac{1}{24}\frac{\text{O.S.}(f, \Pi_i)}{\delta}.$$

From (11) and (13),

$$\text{O.S.}(f, \Pi_i) \geqslant \epsilon$$

and so the variation of f on S_i is at least

$$\frac{\epsilon}{24\delta}.$$

Now, by (12), the least number in S_i is greater by at least δ_1 than the greatest in S_{i-1}. Since $\delta_1 > \delta$, the union of all the sequences S_i, $i \geqslant N$, is itself a δ-separated sequence, and the variation of f on it is infinite, which contradicts the hypothesis that $f \in$ BCV.

THEOREM 3. *If $f \in IR$ then f is simply integrable if and only if it is of* BCV.

Proof. If $f \in$ BCV and ϵ is any positive number, let δ_1 be the $\delta(f, \epsilon/48)$ and B the $B(f, \epsilon/48)$ of Lemma 2. Since $f \in IR$, there is a positive number δ_2 such that if Π is a partition of $[0, B+1]$ of gauge less than δ_2, $\text{O.S.}(f, \Pi) < \epsilon/2$. Set $\Delta = \min(\delta_1, \delta_2, .01)$. Then if $b \leqslant B+1$ and Π is any partition of $[0, b]$ of gauge less than Δ, $\text{O.S.}(f, \Pi) < \epsilon/2$. If $b > B+1$ and Π is a partition of $[0, b]$ with $|\Pi| < \Delta$, we can write $\text{O.S.}(f, \Pi) = \Sigma_1 + \Sigma_2$, where Σ_1 is the part of the oscillation sum involving subintervals (of the partition) that meet $[0, B]$ and Σ_2 is the remainder. Then $\Sigma_1 < \epsilon/2$. Let b' be the rightmost endpoint of the subintervals of Π that meet $[0, B]$. Σ_2 is the oscillation sum of a partition of $[b', b]$ whose gauge is less than Δ, and $b - b' \geqslant .99 > 12\Delta$. By Lemma 1 there is a Δ-separated sequence $x_0 < x_1 < \cdots < x_n$ in $[b', b]$ with

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \geqslant \frac{\Sigma_2}{24\Delta}. \tag{15}$$

But from the definition of δ_1 and B, the sum in (15) is less than $\epsilon/(48\Delta)$. It follows that $\Sigma_2 < \epsilon/2$. Thus $\text{O.S.}(f, \Pi) < \epsilon$ whenever $|\Pi| < \Delta$, and so f satisfies part 1.) of the Uniform Riemann Condition. Since $f \in IR$, it satisfies part 2.), and so is simply integrable, by Theorem 2.

Conversely, if f is simply integrable and ϵ is any positive number, let $S = \{x_0, x_1, \ldots, x_n\}$ be any ϵ-separated sequence of non-negative numbers. By the Uniform Riemann Condition, there is a $\delta > 0$ such that whenever Π is a partition of any interval $[0, b]$ with $|\Pi| < \delta$, then $\text{O.S.}(f, \Pi) < 1$. Set $\delta' = \min\{\delta, \epsilon\}$. We can add points to the sequence S, inserting each new point midway between points already present, to form a partition Π: $x_0 = y_0 < y_1 < \cdots < y_m = x_n$ that is $\delta'/2$-separated and whose gauge is $\leqslant \delta'$.

If we let M_i and m_i denote, respectively, the sup and inf of f on $[y_{i-1}, y_i]$, then

$$
\begin{aligned}
1 \geqslant \text{O.S.}(f, \Pi) &= \sum_{i=1}^{m} (M_i - m_i)(y_i - y_{i-1}) \\
&\geqslant \frac{\delta'}{2} \sum_{i=1}^{m} (M_i - m_i) \\
&\geqslant \frac{\delta'}{2} \sum_{i=1}^{m} |f(y_i) - f(y_{i-1})| \\
&\geqslant \frac{\delta'}{2} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.
\end{aligned}
$$

Thus the variation of f on S is no greater than $2/\delta'$. Since S was an arbitrary ϵ-separated sequence, we see that $V_\epsilon(f) \leqslant 2/\min\{\delta, \epsilon\}$ for every $\epsilon > 0$; so f is of BCV.

We conclude with a representation of real-valued functions of BCV by functions having a monotonicity property:

Definition. Let ϵ be a positive number. A real-valued function f, defined on a set I of real numbers, is "ϵ-increasing" on I if $f(y) \geqslant f(x)$ whenever x and y are points of I with $y \geqslant x + \epsilon$.

Theorem 4. *Let f be a real-valued function defined on* $[0, \infty)$ *and bounded on every finite subinterval. Then, for every $\epsilon > 0$, f is a difference of two ϵ-increasing functions (on* $[0, \infty)$*). f is of* BCV *if and only if it is, for every $\epsilon > 0$, a difference of two bounded ϵ-increasing functions.*

Proof. Given $\epsilon > 0$, if x is any positive number let Π: $0 = x_0 < x_1 < \cdots < x_n = x$ be an ϵ-separated partition of $[0, x]$. Let $p(x)$ be the sum of those $f(x_r) - f(x_{r-1})$ which are positive, $-n(x)$ the sum of those which are negative. Then:

$$f(x) - f(0) = p(x) - n(x),$$

and

$$\sum_{r=1}^{n} |f(x_r) - f(x_{r-1})| = p(x) + n(x)$$
$$= 2p(x) - f(x) + f(0)$$
$$= 2n(x) + f(x) - f(0).$$

Let $V_\epsilon(x), P_\epsilon(x)$, and $N_\epsilon(x)$ be, respectively, the suprema of $\Sigma\,|\,f(x_r) - f(x_{r-1})|$, $p(x)$, and $n(x)$ over all ϵ-separated partitions of $[0, x]$. Then

$$V_\epsilon(x) = 2P_\epsilon(x) - f(x) + f(0) = 2N_\epsilon(x) + f(x) - f(0)$$

and so

$$f(x) = (P_\epsilon(x) + f(0)) - N_\epsilon(x).$$

Both $N_\epsilon(x)$ and $P_\epsilon(x)$ (and so also $P_\epsilon(x) + f(0)$) are ϵ-increasing functions of x on $[0, \infty)$ since any ϵ-separated partition of $[0, x]$ can be extended to an ϵ-separated partition of $[0, y]$ if $y \geqslant x + \epsilon$. Thus the first part of the theorem is proven.

If $f \in$ BCV, $V_\epsilon(f)$ is finite, and $V_\epsilon(x) \leqslant V_\epsilon(f)$ for every x. So V_ϵ is bounded, and since P_ϵ and N_ϵ are non-negative and $P_\epsilon(x) + N_\epsilon(x) = V_\epsilon(x)$, both P_ϵ and N_ϵ are bounded. Conversely, if for each $\epsilon > 0$ both P_ϵ and N_ϵ are bounded, let $x_0 < x_1 < \cdots < x_r$ be any ϵ-separated sequence in $[0, \infty)$. Then

$$f(x_r) - f(x_{r-1}) = P_\epsilon(x_r) - N_\epsilon(x_r) - P_\epsilon(x_{r-1}) + N_\epsilon(x_{r-1})$$
$$= (P_\epsilon(x_r) - P_\epsilon(x_{r-1})) - (N_\epsilon(x_r) - N_\epsilon(x_{r-1})).$$

Therefore

$$|\,f(x_r) - f(x_{r-1})| \leqslant (P_\epsilon(x_r) - P_\epsilon(x_{r-1})) + (N_\epsilon(x_r) - N_\epsilon(x_{r-1})),$$

since each of the two terms on the right is non-negative; and

$$\sum_{r=1}^{n} |f(x_r) - f(x_{r-1})| \leqslant P_\epsilon(x_n) - P(x_0) + N_\epsilon(x_n) - N_\epsilon(x_0).$$

Therefore $V_\epsilon(f)$ is less than or equal to twice the bound of P_ϵ plus twice the bound of N_ϵ, and $f \in$ BCV.

Appendix

A proof that no sequence of quadrature formulas of the form (1) converges for all $f \in IR$: Assume that Q_n is as in (1), and that $\lim_{n\to\infty} Q_n(f) = \int_0^\infty f$ for every f in IR.

If a and b are non-negative numbers, $a < b$, let f be the characteristic function of the interval $[a, b]$. $Q_n(f)$ is just the sum of those $a_{r,n}$ for which $x_{r,n} \in [a, b]$. This sum must approach $b - a$ as $n \to \infty$; so some of those $a_{r,n}$ must be positive if n is sufficiently large.

Now choose n_1 so large that at least one of the a_{r,n_1}—say a_{r_1,n_1}—is positive. Choose ϵ_1 so that the interval $I_1 = (x_{r_1,n_1} - \epsilon_1, x_{r_1,n_1} + \epsilon_1)$ contains no x_{r,n_1} other than x_{r_1,n_1}, and also so that $\epsilon_1/a_{r_1,n_1} < 1/8$. Let $g_1(x)$ be zero outside I_1, equal to $1/a_{r_1,n_1}$ at x_{r_1,n_1}, and linear on the two intervals $[x_{r_1,n_1} - \epsilon_1, x_{r_1,n_1}]$ and $[x_{r_1,n_1}, x_{r_1,n_1} + \epsilon_1]$. Then $Q_{n_1}(g_1) = 1$ and $\int_0^\infty g_1 < 1/8$. Now choose n_2 and r_2 so that $x_{r_2,n_2} > x_{n_1,n_1} + 1$, $a_{r_2,n_2} > 0$. If $Q_{n_2}(g_1) \geqslant 1$, set $g_2(x) \equiv 0$. Otherwise write $\alpha = 1 - Q_{n_2}(g_1)$ and choose ϵ_2 so small that $I_2 = (x_{r_2,n_2} - \epsilon_2, x_{r_2,n_2} + \epsilon_2)$ contains no x_{r,n_2} other than x_{r_2,n_2} and $\alpha\epsilon_2/a_{r_2,n_2} < 1/16$. Let $g_2(x)$ be zero outside I_2, equal to $\alpha/a_{r_2,n_2}$ at x_{r_2,n_2}, and linear on the intervals $[x_{r_2,n_2} - \epsilon, x_{r_2,n_2}]$ and $[x_{r_2,n_2}, x_{r_2,n_2} + \epsilon]$. Then $Q_{n_1}(g_1 + g_2) = Q_{n_1}(g_1) \geqslant 1$, $Q_{n_2}(g_1 + g_2) \geqslant 1$, and $\int_0^\infty (g_1 + g_2) < 3/16$. Continuing thus we produce an infinite sequence $n_1, n_2, \ldots$, and an infinite sequence of functions $g_1, g_2, \ldots$, such that:

1.) $g = g_1 + g_2 + \cdots$ is in IR,

2.) $\int_0^\infty g \leqslant 1/2$,

3.) $Q_{n_i}(g) \geqslant 1, \qquad i = 1, 2, \ldots .$

This contradicts our assumption.

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